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SNAP-BUCKLING OF SHELL-TYPE STRUCTURES UNDER STOCHASTIC LOADING

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Abstract—The snap-buckling instability of shallow shell-type structures that are subjected to a random transverse load is presented. The deformation of the structure is primarily under a symmetric and an antisymmetric mode and the investigation employs a method initially proposed by Kramers in the theory of kinetics of chemical reactions and later adapted by the present authors for the case of symmetric snap-through of shallow, two pinned arches. Analytical expressions are derived for the probability of snap-buckling in a time interval T in terms of the potential energy functions in the neighbourhood of the stable and unstable equilibrium states of the structure.

INTRODUCTION

PROBLEMS encountered in the theory of stability of elastic structures are basically of two types, (i) bifurcation, and (ii) snap-through. A large number of investigations have been carried out on such problems when the loading is static. Corresponding stability investigations for dynamic loading are known as parametric instability and dynamic snap-through problems. Parametric instability is the case of an initially straight prismatic column whose ends are hinged and subjected to a periodic, axial, compressive force whereas in dynamic snap-through the instability phenomenon involves a structure, such as a shallow shell under a transverse load, leaving an initial stable equilibrium configuration for another stable equilibrium position undergoing a finite jump over a potential barrier at a certain critical value of the load. The latter is analogous to the "jump phenomenon" in the theory of nonlinear vibrations and has been extensively investigated by Mettler [1], Hoff and Bruce [2], Lock [3], etc. However, in modern aircraft and missiles the excitation experienced by the structure cannot be adequately described in terms of deterministic functions of time. These forces fluctuate in a random manner over a wide band of frequencies and have to be considered as stochastic functions of time defined only in probabilistic terms. Hence, an examination of the dynamic stability problems by a statistical approach is necessary as well as realistic.

Investigations on the instability of elastic structures under stochastic excitations have started only recently. The treatment of the parametric instability under random loads is available in papers by Samuels [4], Ariaratnam [5] and Stratanovich *et al.* [6]. Corresponding investigations in the case of snap-through problems are very few. Vorovich [7]

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examined the snap-through of a cylindrical panel under stochastic edge compression and obtained the stationary probability distribution of the response which is valid only for infinite time lapses from the given initial equilibrium state of the structure. Goncharenko [8] studied the same problem but obtained a quasi-stationary solution of the stochastic differential equation of the response. In a previous paper the present authors [9] considered stochastically loaded shallow, two-pinned arches and derived the probability of first symmetric snap-through in a finite time T from "locally" stationary solution of the Fokker–Planck equation. The principal idea of their analysis is derived from Kramer's study [10] of chemical reactions where a method is indicated for calculating the probability of escape of particles over a potential barrier through the shuttling action of Brownian forces caused by a surrounding medium in temperature equilibrium. This paper is an extension of Kramer's analysis for snap-buckling cases where symmetric and antisymmetric modes are involved in the buckling process.

A shell-type structure whose initial rise exceeds a certain specific value exhibits a different buckling behaviour from that with a very small initial central height. The latter loses its stability under a single symmetric mode of deformation and is generally known as a "snap-through" problem, while the former deforms under a symmetric and an antisymmetric mode displaying a "snap-buckling" instability at a critical loading condition. In the snap-through case discussed by Goncharenko [8] and the present authors [9], the potential energy function has a single local maximum corresponding to an unstable equilibrium configuration as shown in Fig. 1; and when the structure is subjected to a stochastic load the problem involves the determination of the rate of diffusion of the ensemble probability density across the potential hump in a two-dimensional phase space. In the problem of snap-buckling the potential energy surface has a saddle point representing the unstable equilibrium state of the structure and when the structure is under stochastic loads, the problem would then involve a discussion of the diffusion of the probability density along a certain principal direction in a four-dimensional phase space. Such a method is discussed in this paper. The analysis presented is confined to two degrees of freedom systems, that is for structures exhibiting primarily two modes of deformation, namely a symmetric and an antisymmetric mode.

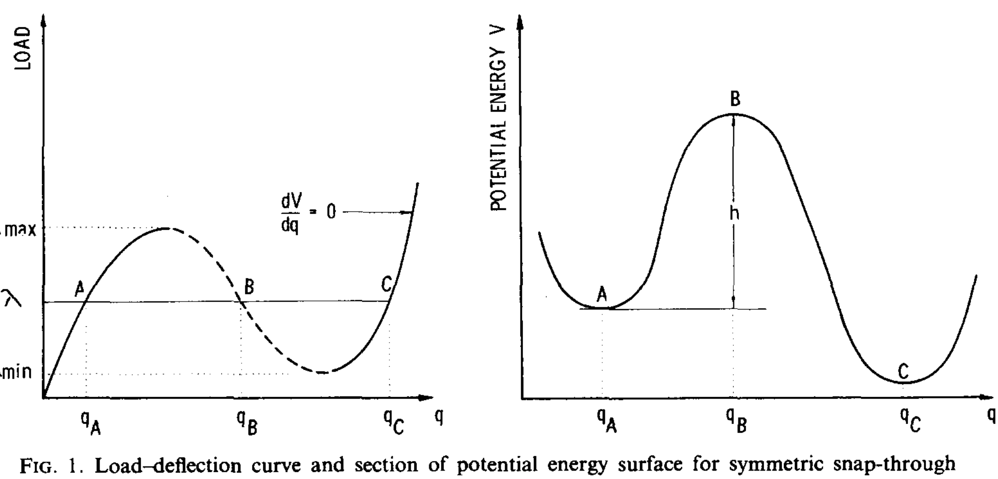

FIG. 1. Load–deflection curve and section of potential energy surface for symmetric snap-through problem.

FORMULATION OF THE PROBLEM

Consider a shallow shell-type structure (for example arch, curved panel, shell, etc.) whose initial geometry is such that primarily two modes of deformation are excited under the applied loads. The structure is originally in equilibrium under a symmetrically distributed static load of known intensity λ. At time $t = 0$, suppose the structure is subjected to an additional distributed stochastic load of intensity $\xi(t)$ having zero mean value. Let q_1 and q_2 be the amplitudes of the first symmetric and antisymmetric modes of deformation. Assuming that the applied loads have only a q_1-component and no q_2-component (which is the case in many structural problems), and that the coefficient of damping β is the same in both modes of deformation, the equations of motion of the structure may be written in the following general form

$$\begin{aligned} \ddot{q}_1+\beta\dot{q}_1-F_1(q_1,q_2,\lambda) &= \xi(t), \\ \ddot{q}_2+\beta\dot{q}_2-F_2(q_1,q_2,\lambda) &= 0, \end{aligned} \tag{1}$$

where F_1 and F_2 are the static force fields along q_1 and q_2 coordinate directions and are related to the potential function $V(q_1, q_2, \lambda)$ of the structure by

$$\begin{aligned} F_1 &= -\frac{\partial V}{\partial q_1}, \\ F_2 &= -\frac{\partial V}{\partial q_2}. \end{aligned} \tag{2}$$

For any arbitrary but real value of the conservative load parameter λ, the possible equilibrium configurations of the structure can be obtained by solving the following equations:

$$\begin{aligned} \frac{\partial V}{\partial q_1} &= 0, \\ \frac{\partial V}{\partial q_2} &= 0. \end{aligned} \tag{3}$$

Figure 2 shows the static load–deflection curves of the structure described by the above equations. Let λ_{max} correspond to the static buckling load of the structure. For values of

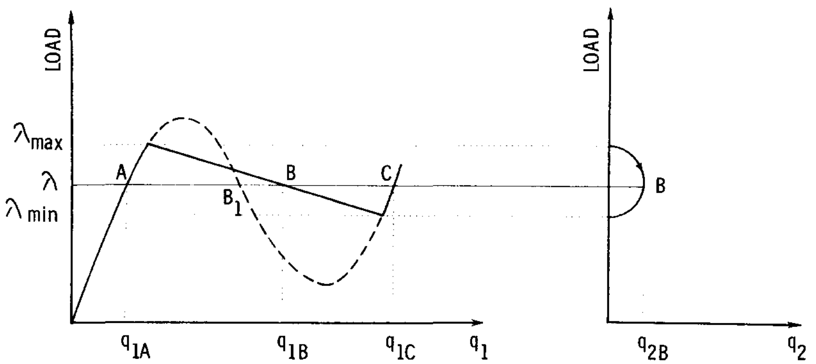

FIG. 2. Load–deflection curves for the snap-buckling problem.

$\lambda < \lambda_{max}$, the deformation of the structure is essentially symmetric. When $\lambda = \lambda_{max}$, an antisymmetric mode of deformation is picked up and the structure starts to buckle. The configuration of the structure after snapping is again symmetric. It may be noted that such a buckling phenomenon is initiated at a point where the original equilibrium path of the structure branches into another equilibrium path at the onset of an antisymmetric deformation. Hence the terminology "snap-buckling".

For any value of the load parameter λ satisfying the inequality $\lambda_{min} < \lambda < \lambda_{max}$, where λ_{min} is the lower critical load (that is the buckling load achieved when the loading is progressively reduced after the structure has snapped), there are four distinct equilibrium states of the structure as indicated by points A, B, B_1 and C in Fig. 2. The contours of equal potential energy on the q_1–q_2 plane are shown in Fig. 3. Here, the equilibrium configurations of the structures are located at the bottom of the "depressions", the top of the "hills" and

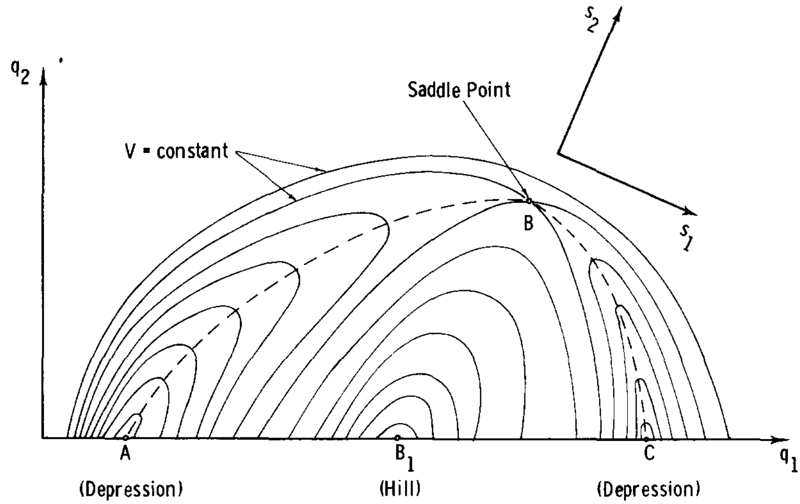

FIG. 3. Contours of equal potential energy.

the "saddle points". The initial equilibrium configuration at A and the buckled configuration at C are associated with the symmetric mode q_1 only and are stable because they are located at the bottom of depressions in the potential energy surface. The equilibrium configuration at the point B_1 is described by the symmetric deformation q_1 only and is unstable since it is situated at the top of a hill in the potential energy surface. If the equilibrium path of the structure passes through the point B_1, only a "snap-through" type of instability will be initiated; then, the probability of snapping under stochastic loading for such structures will be given by the analysis presented in paper [9]. The equilibrium configuration of the structure at the point B is associated with both symmetric and antisymmetric deformations and is unstable since it is located at a saddle point of the potential energy surface.

The dynamic path followed by the structure in the q_1–q_2 configuration space, starting from the initial stable equilibrium configuration corresponding to the point A, will depend on the nature of the perturbation given. If the magnitude of the perturbation is sufficiently small, it is reasonable to expect that the dynamic path from the point A to the vicinity of the point C will tend to follow one for which the slope of ascent is a minimum and the slope of descent is a maximum everywhere on the potential energy surface. Such a path can be defined analytically if the perturbation is a deterministic function of time [2].

Initially at $t = 0$, the structure is in equilibrium in the stable configuration ($q_1 = q_{1A}$, $q_2 = 0$) corresponding to the load parameter λ, when the stochastic excitation $\xi(t)$ is applied. The dynamic state of the structure at any subsequent time t can then be described by the phase variables (q_1, q_2, v_1, v_2) where $v_1 = \dot{q}_1$ and $v_2 = \dot{q}_2$. The motion of a point or a "particle" described by the above variables in the phase space is random since $\xi(t)$ is stochastic. That is, for different members of the ensemble of random loading $\xi(t)$, there will be an ensemble of trajectories of the phase point or "particle" starting from the position $(q_{1A}, 0, 0, 0)$ corresponding to the point A in the phase space. During a time interval T, some of these paths may have remained entirely within the neighbourhood of the initial equilibrium point A while some others may have surmounted the potential barrier and reached the vicinity of the point C which indicates a buckled state of the structure. The problem is to determine the probability P_T that in a time interval T, starting from an initial stable configuration at A, the structure would have snap-buckled and be found in the neighbourhood of the configuration at C.

Since the potential energy difference between the equilibrium points A and B is less than that between A and B_1, it may be expected that a considerable fraction of the paths of the phase points leading towards the point C will be clustered in the neighbourhood of the point B as indicated in Fig. 4; and the projection of these trajectories on the $q_1 - q_2$ plane will be nearly parallel, around the point B, to the projection of the line of downward curvature of the potential energy surface at the saddle point B. Let s_1 and s_2 denote the lines of principal curvature coordinates of the potential energy surface and suppose that the s_1 coordinate corresponds to the direction of the downward curvature at the saddle point B. It will be shown in the following pages that the probability P_T of first snap-buckling may be evaluated from a steady flux rate j_B of the probability density across the hypersurface $s_1 = s_{1B}$ in the phase space. To facilitate the evaluation of this flux rate j_B, it is then convenient to reformulate the problem in terms of the principal s_1 and s_2 coordinates.

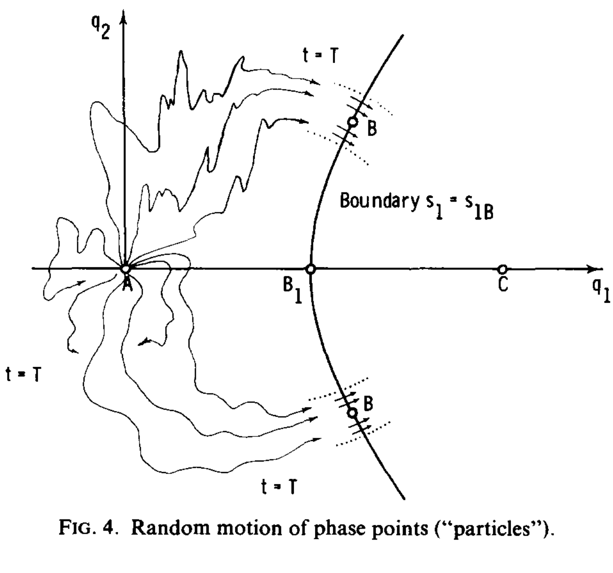

FIG. 4. Random motion of phase points ("particles").

FOKKER–PLANCK EQUATION

When the stochastic load $\xi(t)$ is applied, the state of the structure at any instant is described by the phase variables (q_1, q_2, v_1, v_2, t). If the first probability density $p(q_1, q_2, v_1, v_2, t)$ of the vector random process is known, the probability of first snap-buckling will be given by

$$P_T = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{q_{1B}}^{\infty} p(v_1, v_2, q_1, q_2, T)\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}v_1\, \mathrm{d}v_2. \tag{4}$$

Here it is assumed that the trajectories describing the random motion of the phase point end up in the neighbourhood of the depression at C once they cross the boundary passing through B. Obviously, the probability density $p(v_1, v_2, q_1, q_2, t)$ must satisfy the initial conditions, and the time interval T should be sufficiently small compared to the relaxation time τ, so that no trajectory that has once crossed over the boundary $q_1 = q_{1B}$ has drifted back and the expression (4) may represent the probability of first snapping of the structure.

If $\xi(t)$ is a stationary, wide-band random process with a delta correlation, the response process (v_1, v_2, q_1, q_2, t) can be approximated by a Markov Process [11] and the first probability density $p(v_1, v_2, q_1, q_2, t)$ may be described by the Fokker–Planck equation. Writing the equations of motion (1) as a set of state equations

$$\dot{q}_1 = v_1, \qquad \dot{v}_1 = -\beta v_1 + F_1 + \xi(t),$$
$$\dot{q}_2 = v_2, \qquad \dot{v}_2 = -\beta v_2 + F_2,$$

the Fokker–Planck equation giving the probability of the response is

$$\frac{\partial p}{\partial t} = -F_1\frac{\partial p}{\partial v_1} - v_1\frac{\partial p}{\partial q_1} - F_2\frac{\partial p}{\partial v_2} - v_2\frac{\partial p}{\partial q_2}$$
$$+\beta\frac{\partial}{\partial v_1}\left(v_1 p + \frac{D}{\beta}\frac{\partial p}{\partial v_1}\right) + \beta\frac{\partial}{\partial v_2}(v_2 p), \tag{5}$$

where

$$\langle \xi(t_1)\xi(t_2)\rangle = 2D\delta(t_1 - t_2),$$

angular brackets denoting the ensemble average and D, the intensity coefficient.

Any solution of (5) must satisfy the condition

$$p(v_1, v_2, q_1, q_2, t) \to \delta(q_1 - q_{1A})\delta(q_2)\delta(v_1)\delta(v_2) \qquad \text{for } t \to 0^+, \tag{6}$$

which expresses the fact that the structure is initially in equilibrium at the point $A(q_1 = q_{1A}, q_2 = 0)$.

DIFFUSION OF PROBABILITY IN THE PHASE SPACE

An exact solution of the equation (5) satisfying the boundary condition (6) is not known. But, if the energy imparted by the stochastic load $\xi(t)$ is assumed small in comparison with the height of the potential barrier, that is, if

$$\frac{D}{\beta} \ll h, \qquad h = V(q_{1B}, q_{2B}) - V(q_{1A}, 0), \tag{7}$$

then the equation (5) can be approximately solved. To visualize this, consider the diffusion flow of the phase points that represent the dynamical states of a large number of identical structures each subjected to the action of a different sample function of the random process $\xi(t)$. The Fokker–Planck equation (5) may then be taken to be the continuity equation for the diffusion of these phase points or "particles" and the probability density p may be interpreted as the mass density for this flow. At time $t = 0$, there is a concentration of particles near the point A given by the delta function distribution (6). With the passage of time, the diffusion of particles from the point A towards the point C primarily through the transition point B will be started tending to establish a statistical equilibrium for the flow. Under the assumption (7), the diffusion will be slow and may be considered as a quasi-stationary process. Further, if the time of interest T is much smaller than the correlation time τ, the flow will be principally towards the point C. The probability density distribution in the neighbourhood of any point in the phase space will then be given by the stationary solution of the diffusion equation (5) with the potential function V appropriate to that neighbourhood indicating that the diffusion process is locally stationary. The probability density in the neighbourhood of A will be the stationary solution of the equation (5) with the antisymmetric deformation $q_2 = 0$. That is,

$$p_A = \alpha_A \exp\left\{-\frac{\beta}{D}\left[\frac{1}{2}v_1^2+\frac{1}{2}(V_{11})_A(q_1-q_{1A})^2-h\right]\right\}, \tag{8}$$

where α_A is a normalizing constant and $(V_{11})_A$ is the quantity d^2V/dq_1^2 evaluated at the point A. The value of the potential function is chosen such that it is zero at the point B. In the neighbourhood of the point B, the potential energy is a function of both the symmetric and antisymmetric deformations, and is given by the expansion

$$V_{\text{near}\,B} \simeq \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(q_i-q_{iB})(q_j-q_{jB})\left(\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right)_{\text{at}\,B}.$$

The presence of the terms $\partial^2V/\partial q_i\,\partial q_j$ (for $i \neq j$) in the Fokker–Planck equation valid near B poses certain difficulties in obtaining a desired type of solution. Also, the flux rate j_B of the probability density across a hyperplane through the saddle point B is in the direction of the downward curvature of the potential energy surface at B, as explained before. Therefore, the Fokker–Planck equation governing the probability density of the particles in the neighbourhood of the point B must be set up in terms of the lines of curvature coordinates of the potential energy surface at the point B.

As mentioned previously, let s_1 and s_2 represent the new axes of reference with respect to which the motion of the structure, in the vicinity of the equilibrium point B, will be now described. The transformation law relating the s-system to the original q-system of coordinates is given by

$$\begin{aligned} q_1 &= s_1\cos\theta+s_2\sin\theta,\\ q_2 &= -s_1\sin\theta+s_2\cos\theta, \end{aligned} \tag{9}$$

where θ is the angle of transformation. Let V^* represent the potential energy of the structure in terms of the new variables s_1 and s_2. The value of $V^*(s_1, s_2, \lambda)$ can be calculated from $V(q_1, q_2, \lambda)$ using the relations (9). By the definition of the s-axes, the variation of V^* occurs uniquely with respect to each s_1 and s_2 direction and the cross differential terms $\partial^2V^*/\partial s_i\,\partial s_j$ vanish for $i \neq j$.

It may be noted that the saddle point B represents a "depression" with respect to the principal s_2-direction and a "hill" with respect to the principal s_1-direction in the potential energy contours shown in Fig. 3. This means that the cluster of random phase trajectories in the neighbourhood of the point B will be essentially directed along the s_1-direction towards the point C as shown in Fig. 4. Hence, the flux rate j_B of the particles across the boundary $s_1 = s_{1B}$ must be obtainable from the marginal probability density distribution $p_1(s_1, \dot{s}_1)$. Explicitly stated, j_B is given by

$$\begin{aligned} j_B &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1 p(s_{1B}, s_2, u_1, u_2)\, \mathrm{d}s_2\, \mathrm{d}u_1\, \mathrm{d}u_2, \\ &= \int_{-\infty}^{\infty} u_1 p_1(s_{1B}, u_1)\, \mathrm{d}u_1, \end{aligned} \tag{10}$$

where $u_1 = \dot{s}_1, u_2 = \dot{s}_2$.

The equations of motion of the structure (1), in the neighbourhood of the equilibrium point B, referred to the principal s-axes become

$$\begin{aligned} \ddot{s}_1 + \beta\dot{s}_1 - F_1^*(s_1, \lambda) &= \xi(t)\cos\theta, \\ \ddot{s}_2 + \beta\dot{s}_2 - F_2^*(s_2, \lambda) &= \xi(t)\sin\theta, \end{aligned} \tag{11}$$

where $F_1^* = -\partial V^*/\partial s_1, F_2^* = -\partial V^*/\partial s_2$. The marginal probability distributions $p_1(s_1, u_1)$ and $p_2(s_2, u_2)$, in the neighbourhood of the point B, are then given by the stationary solution of the following independent Fokker–Planck equations:

$$0 = -F_1^*\frac{\partial p_1}{\partial u_1} - u_1\frac{\partial p_1}{\partial \bar{s}_1} + \beta\frac{\partial}{\partial u_1}\left(u_1 p_1 + \frac{D\cos^2\theta}{\beta}\frac{\partial p_1}{\partial u_1}\right), \tag{12}$$

$$0 = -F_2^*\frac{\partial p_2}{\partial u_2} - u_2\frac{\partial p_2}{\partial \bar{s}_2} + \beta\frac{\partial}{\partial u_2}\left(u_2 p_2 + \frac{D\sin^2\theta}{\beta}\frac{\partial p_2}{\partial u_2}\right). \tag{13}$$

where $\bar{s}_1 = s_1 - s_{1B}, \bar{s}_2 = s_2 - s_{2B}$.

RATE OF FLUX OF PROBABILITY DENSITY

The marginal probability densities $p_1(s_1, u_1)$ and $p_2(s_2, u_2)$ in the neighbourhood of the point B are given by the solution of the equations (12) and (13) respectively with the potential energy V^* appropriate to that neighbourhood given by the expansion

$$V^*_{\text{near } B} \simeq \tfrac{1}{2}(V^*_{11})_B\bar{s}_1^2 + \tfrac{1}{2}(V^*_{22})_B\bar{s}_2^2,$$

where $(V^*_{ii})_B$ represents the value $\partial^2 V^*/\partial s_i^2$ calculated at the point B. The quantities $(V^*_{11})_B$ and $(V^*_{22})_B$ can be evaluated from $(V_{11})_B$, $(V_{22})_B$ and $(V_{12})_B$ using a Mohr's circle transformation:

$$(V^*_{11})_B = \frac{(V_{11})_B + (V_{22})_B}{2} - \frac{1}{2}\{[(V_{11})_B - (V_{22})_B]^2 + 4(V_{12})_B^2\}^{1/2}, \tag{14}$$

$$(V^*_{22})_B = \frac{(V_{11})_B + (V_{22})_B}{2} + \frac{1}{2}\{[(V_{11})_B - (V_{22})_B]^2 + 4(V_{12})_B^2\}^{1/2}, \tag{14a}$$

$$(V^*_{12})_B = (V^*_{21})_B = 0. \tag{14b}$$

The angle θ of the transformation is given by the equation

$$\tan 2\theta = \frac{2(V_{12})_B}{(V_{11})_B - (V_{22})_B}. \tag{14c}$$

Using the above results in equations (12) and (13), the densities p_1 and p_2 are found to be

$$p_1(s_1, u_1) = \alpha_1 \exp\left[\frac{-\beta}{2D\cos^2\theta}(u_1^2 + (V_{11}^*)_B \bar{s}_1^2)\right], \tag{15}$$

$$p_2(s_2, u_2) = \alpha_2 \exp\left[\frac{-\beta}{2D\sin^2\theta}(u_2^2 + (V_{22}^*)_B \bar{s}_2^2)\right], \tag{16}$$

where α_1 and α_2 are normalizing constants. The solutions (15) and (16) correspond to a completely stationary situation with a net flux across any boundary through the point B equal to zero. However, due to the negative sign for V_{11}^* representing an unstable equilibrium state at the point B, another solution for $p_1(s_1, u_1)$ is possible for which $\alpha_1 = \alpha_1(z)$, $z = u_1 - c\bar{s}_1$ where c is a constant to be determined. Since $(V_{22}^*)_B$ is positive, the only solution possible is the one given in (16). This means that there is a diffusion of the probability current taking place only along the principal s_1 direction, while along the principal s_2 direction the net flux is zero. Writing $\Phi_1 = -(V_{11}^*)_B$ and seeking a solution of the form

$$p_1 = \alpha_1(z) \exp\left[\frac{-\beta}{2D\cos^2\theta}(u_1^2 - \Phi_1 \bar{s}_1^2)\right], \tag{17}$$

it may be seen from the Fokker–Planck equation (12) that

$$(cu_1 - \beta u_1 - \Phi_1 \bar{s}_1)\frac{d\alpha_1}{dz} + D\cos^2\theta \frac{d^2\alpha_1}{dz^2} = 0.$$

The above equation is valid only if c satisfies the condition

$$(c-\beta)u_1 - \Phi_1 \bar{s}_1 = (c-\beta)z,$$

or

$$c = \frac{\beta}{2} \pm \left(\frac{\beta^2}{4} + \Phi_1\right)^{1/2}. \tag{18}$$

$\alpha_1(z)$ is then given by

$$z(c-\beta)\frac{d\alpha_1}{dz} + D\cos^2\theta \frac{d^2\alpha_1}{dz^2} = 0,$$

and is found to be

$$\alpha_1(z) = A_1 \int^z \exp\left[\frac{-(c-\beta)z^2}{2D\cos^2\theta}\right] dz, \tag{19}$$

where A_1 is a constant. The density distribution (17) will be bounded only if $c > \beta$; hence, the positive sign is chosen in the expression (18). If the lower limit of the integral in (19) is taken as $-\infty$, then the expression (17) will give a distribution for which $p_1 \to 0$ for $s_1 \gg s_{1B}$ (in the region near to and right of the point C) and $p_1 \to p_A$ for $s_1 \ll s_{1B}$ (for regions near to and left of the point A). Hence, p_1 may be considered as the asymptotic form valid near B

of a stationary marginal distribution

$$p = \left\{A_1 \int_{-\infty}^{u_1 - c\bar{s}_1} \exp\left[\frac{-(c-\beta)z^2}{2D\cos^2\theta}\right] dz\right\} \exp\left[\frac{-\beta}{2D\cos^2\theta}(u_1^2 + (V_{11}^*)_B \bar{s}_1^2)\right]. \tag{20}$$

The quantity j_B may now be calculated from the expression (10).

$$\begin{aligned} j_B &= \int_{-\infty}^{\infty} u_1 p_1(s_{1B}, u_1)\, du_1, \\ &= \int_{-\infty}^{\infty} \left\{A_1 \int_{-\infty}^{u_1} \exp\left[\frac{-(c-\beta)z^2}{2D\cos^2\theta}\right] dz\right\} u_1 \exp\left[\frac{-\beta}{2D\cos^2\theta} u_1^2\right] du_1, \\ &= A_1 \frac{D\cos^3\theta}{\beta}\left(\frac{2D\pi}{c}\right)^{1/2}, \end{aligned} \tag{21}$$

where $c = \beta/2 + (\beta^2/4 + \Phi_1)^{1/2}$.

PROBABILITY OF FIRST SNAP-BUCKLING

The number of particles in the neighbourhood of the equilibrium point A is given by the stationary probability density p_A in the expression (8). Since $p_1 \to p_A$ in the neighbourhood of A, $\alpha_1(z)$ must tend to α_A. Therefore, from the expression (19)

$$\begin{aligned} \alpha_A &= A_1 \int_{-\infty}^{\infty} \exp\left[\frac{-(c-\beta)z^2}{2D\cos^2\theta}\right] dz, \\ &= A_1 \cos\theta\left(\frac{2D\pi}{c-\beta}\right)^{1/2}. \end{aligned} \tag{22}$$

The number of particles originally near the point A is

$$\begin{aligned} n_A &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_A\, d(q_1 - q_{1A})\, dv_1, \\ &= A_1 \cos\theta\left(\frac{2D\pi}{c-\beta}\right)^{1/2} \frac{2D\pi}{\beta(V_{11})_A^{1/2}} \exp\left(\frac{\beta h}{D}\right). \end{aligned} \tag{23}$$

The probability P_T of first snap-buckling in a time T is

$$P_T = \frac{j_B T}{n_A},$$

and on using the expressions (18), (21) and (23)

$$P_T = \frac{T}{2\pi}\cos^2\theta\left[\frac{(V_{11})_A}{-(V_{11}^*)_B}\right]^{1/2}\left\{\left[\frac{\beta^2}{4}-(V_{11}^*)_B\right]^{1/2}-\frac{\beta}{2}\right\}\exp\left(\frac{-\beta h}{D}\right). \tag{24}$$

DISCUSSION

It may be observed that the expression derived for the probability P_T of first snap-buckling is valid only if the following conditions are satisfied.

(i) The height of the potential barrier is large in comparison with the energy supplied by the random forces as given by the inequality (7), and

(ii) the prescribed time interval T for the instability to occur is small compared to the relaxation time τ of the diffusion process so that the probability P_T evaluated represents the first rather than the eventual snap-buckling. Though an exact estimate of τ requires the complete nonstationary solution of the diffusion equation (5) which is not available at present, it may be seen that the relaxation time decreases with an increase in the rate of diffusion, that is, it decreases with an increase in the intensity coefficient D of the process $\xi(t)$.

The value of the probability P_T may be expected to be influenced considerably by the term exponential $(-\beta h/D)$ in the expression (24). Hence, it is reasonable to expect that the probability of snap-buckling will be highly sensitive to any small variation in the static load parameter λ (which will cause a large variation in the height h of the potential barrier and hence affecting greatly the exponential term) and in the coefficient of damping β. It may also be noted that for cases where the antisymmetric deformation q_2 is not excited, the formula (24) reduces to the one obtained by the authors in their previous paper [9] where a symmetric snap-through case of shallow arches has been presented. For such problems where $q_2 = 0$, the principal s_1 coordinate coincides with the q_1 coordinate with the angle θ of transformation in expression (9) being zero. Consequently, the saddle point B in Fig. 3 coincides with the equilibrium point B_1 indicating a snap-through type of instability of the structure.

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Абстракт—Дается задача неустойчивости прощелкивания конструкций, типа пологих оболоуек, подверженных действию произвольной поперечной нагрузки. Деформация конструкции находится, сначала, в симметрической и антисимметрической форме. Исследование используем метод, предложенный сперва крамером в теории кинетики химических реакций, и дале адаптированнйй авторами для случая прощелкивания пологих, двух скрепленных арок. Выводятся аналитические зависимости для вероятности прощелкивания для интервала времени T, в выражениях для функций потенциальной энергии, в соседстве устойчивых и неустойчивых состояний равновесия конструкции.